

# ENERGY AND MOMENTUM OF A STATIONARY BEAM OF LIGHT

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## Abstract

The energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg give the same and meaningful results for the energy and momentum of the Bonnor spacetime describing the gravitational field of a stationary beam of light. The results support the Cooperstock hypothesis.

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## I. INTRODUCTION:

Since Einstein proposed the General Theory of Relativity, relativists have not been able to agree upon a definition of the energy-momentum distribution associated with the gravitational field (see Ref. [1] and references therein). In 1990, H. Bondi[2] argued that General Relativity does not permit a non-localizable form of energy, so, in principle, we should expect to be able to find an acceptable definition.

Energy and momentum density are usually defined by a second rank tensor  $T_i^k$ . The conservation of energy and momentum are described by the requirement that the tensor's divergence is zero. However, in General Relativity, the partial derivative in the usual conservation equation  $T_{i,k}^k = 0$  is replaced by a covariant derivative.  $T_i^k$  then represents the energy and momentum of matter and all non-gravitational fields and no longer satisfies  $T_{i,k}^k = 0$ . A contribution from the gravitational field must be added to obtain an energy-momentum expression with zero divergence. Einstein first obtained such an expression and many others such as Landau and Lifshitz, Papapetrou, and Weinberg gave similar prescriptions (see Ref. [3]). The expressions they gave are called energy-momentum complexes because they can be expressed as a combination of  $T_i^k$  and a pseudotensor, which is interpreted to represent the energy and momentum of the gravitational field. These complexes have been heavily criticized because they are non-tensorial, i.e. they are coordinate dependent. For the Einstein, Landau-Lifshitz, Papapetrou, and Weinberg (later ELLPW) energy-momentum complexes, one gets physically meaningful results only in "Cartesian coordinates" (see Ref. [4, 5]). Because of this drawback, many others, including Møller [5], Komar[6], and Penrose[7], have proposed coordinate independent definitions. Each of these, however, has its own drawbacks (for further discussion see Ref. [8]).

Recently, the energy momentum complexes of ELLPW have shown promise and may prove to be more useful than the others. Much work has been produced in the past decade showing the these complexes give meaningful results for many well known metrics ( Ref. [9, 10, 11]). Also, Aguirregabiria *et al.*[12] showed that they coincide for all Kerr-Schild class metrics which include the Schwarzschild, Reissner-Nordström, Kerr-Newman, Vaidya, Dybney *et al.*, Kinnersley, Bonnor-Vaidya, and Vaidya-Patel spacetimes (for references see in [13]).

In this Letter, we will calculate, using the energy momentum complexes of ELLPW, the energy and momentum densities of the metric given by Bonnor[14] describing a stationary beam of light. First we will show that the metric is of Kerr-Schild class, and then we will use the results given by Aguirregabiria *et. al.* to compute its energy and momentum densities. Finally, we will discuss the physical implications of our calculations.

We use the convention that Latin indices take values from 0 to 3 and Greek indices take values from 1 to 3, and take units where  $G = 1$  and  $c = 1$ .

## II. KERR-SCHILD CLASS METRICS

A metric is said to be of Kerr-Schild class if it can be written in the form

$$g_{ab} = \eta_{ab} - 2Vl_al_b \quad (1)$$

where  $\eta$  is the Minkowski metric,  $V$  is a scalar field, and  $l$  is a null, geodesic, and shear free vector field in the Minkowski spacetime. These three properties of  $l$  are respectively expressed as

$$\eta^{ab}l_al_b = 0 \quad (2)$$

$$\eta^{ab}l_{i,a}l_b = 0 \quad (3)$$

$$(l_{a,b} + l_{b,a})l^a_{,c}\eta^{bc} - (l^a_{,a})^2 = 0 \quad (4)$$

Aguirregabiria *et. al.*[12] showed that for general Kerr-Schild class metrics the ELLPW energy-momentum complexes coincide up to raising and lowering of indices by the Minkowski metric. Later, Virbhadra[15] clarified that the vector field  $l$  need only be null and geodesic for the ELLPW complexes to coincide. These energy momentum complexes for any Kerr-Schild class metric are given by

$$\Theta_i{}^k = \eta_{ij}L^{jk} \quad (5)$$

$$L^{ik} = \Sigma^{ik} = W^{ik} = \frac{1}{16\pi}\Lambda^{iklm}_{,lm} \quad (6)$$

where  $\Theta$ ,  $L$ ,  $\Sigma$ , and  $W$  are the Einstein, Landau-Lifshitz, Papapetrou, and Weinberg energy momentum complexes respectively and

$$\Lambda^{ikpq} = 2V(\eta^{ik}l^pl^q + \eta^{pq}l^il^k - \eta^{ip}l^kl^q - \eta^{kq}l^il^p). \quad (7)$$

$\Theta_0^0$ ,  $L^{00}$ ,  $\Sigma^{00}$ , and  $W^{00}$  represent total energy density and  $\Theta_\alpha^0$ ,  $L^{0\alpha}$ ,  $\Sigma^{0\alpha}$ , and  $W^{0\alpha}$  represent momentum density in the  $x^\alpha$  direction.

### III. THE BONNOR METRIC

The Bonnor metric describing a stationary beam of light flowing in the  $Z$  direction is given by the line element[14]

$$ds^2 = -dx^2 - dy^2 - (1 - m)dz^2 - 2mdzdt + (1 + m)dt^2 \quad (8)$$

where  $m$  is a function of  $x$  and  $y$ ,

$$\nabla^2 m = 16\pi\rho \quad (9)$$

$$\rho = -T_3^3 = -T_3^0 = T_0^3 = T_0^0 \quad (10)$$

and  $T_a^b$  is the energy momentum tensor.

The Bonnor metric can be rewritten as

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - m(dt - dz)^2. \quad (11)$$

This is in form required of a Kerr-Schild class metric with

$$V = m/2 \quad (12)$$

$$l_t = 1 \quad (13)$$

$$l_z = -1 \quad (14)$$

Both components of  $l$  are constant so  $l$  is trivially geodesic and shear free. It can be easily shown to be null which proves that the Bonnor metric is of Kerr-Schild class.

We can then easily compute the energy momentum complexes of ELLPW for the metric. We note that, in general  $\Lambda$  exhibits the following symmetries

$$\Lambda^{ikpq} = \Lambda^{kiqp} = \Lambda^{pk iq} \quad (15)$$

Then, we calculate the required, non-vanishing components of  $\Lambda$  to be

$$\Lambda^{0011} = \Lambda^{0022} = m \quad (16)$$

$$\Lambda^{0033} = 2m \quad (17)$$

$$\Lambda^{0311} = \Lambda^{0322} = m \quad (18)$$

$$\Lambda^{0303} = -2m \quad (19)$$

From here we calculate the energy-momentum complexes. The non-vanishing components are, as expected, total energy density and momentum density in the  $z$  direction. These both are given by

$$L^{00} = L^{03} = \frac{1}{16\pi}(m_{,xx} + m_{,yy}) \quad (20)$$

Then, recalling Equations (9) and (10)

$$L^{00} = L^{03} = \rho = T_0^0 = T_0^3 \quad (21)$$

#### IV. DISCUSSION

This result is what would be expected from purely physical arguments. The energy and momentum densities calculated from the complexes of ELLPW coincide and are equal to the energy and momentum density components of  $T_i^k$ .

This simple result supports the Cooperstock hypothesis[16] which states that energy is localized to the region where the energy-momentum tensor is non-vanishing. This hypothesis

would imply that there is no energy-momentum contribution from the "vacuum" regions of spacetime. If true, the hypothesis would have broad implications. For instance, a quantum theory of gravity requires that the gravitons which make up the field carry energy. Also, the hypothesis suggests that gravitational waves have no energy and that current attempts to detect these waves by detecting their energy will be fruitless.

In summary, we have calculated the energy and momentum for a stationary beam of light and found that the results are physically meaningful and support the Cooperstock hypothesis. Certainly the questions of energy and momentum localization in General Relativity as well as the Cooperstock hypothesis are far from resolved, but the Bonnor metric provides one more example where the energy momentum complexes of ELLPW provide, for a physical metric, meaningful results that support the hypothesis.

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